Simulating Jump-Diffusion Processes

Let X be the stochastic process for the log-price of an asset. The dynamics of X are given by the Jump-Diffusion process discussed previously:

$$X_t = \int_0^t \mu_s ds + \int_0^t \sqrt{c_s} dW_s + J_t$$

It is helpful to write it in differential form:

$$dX_t = \underbrace{\mu_t dt + \sqrt{c_t} dW_t}_{\text{diffusion}} + \underbrace{dJ_t}_{\text{jumps}}$$

We will study how to simulate this process. To do so, we break the simulation of X into three parts: simulating the diffusion, simulating the jump process, and combining both. We first assume that the variance is constant, and then deal with the case where it is time varying.

1 Constant Variance

Assume $\sqrt{c_t} \equiv \sigma$ for some $\sigma \in \mathbb{R}$, say $\sigma = 0.011$ (1.10 percent per day). Fix the number of steps per day, say $n = 80$. Then, the discretization interval is $\Delta_n = 1/80$. Also fix the number of days to simulate, $T = 1.25 \times 252$ (a year and 3 months). To simplify, assume the drift is constant and equal to zero ($\mu_t \equiv \mu = 0$).

1.1 The Continuous Part

Denote the diffusion part of the process by $\tilde{X}$. To simulate the diffusion part of the process, generate nT independent draws from the standard normal distribution:

$$Z_i \overset{d}{\sim} \mathcal{N}(0,1) \text{ for } i = 1, 2, \dots, nT$$

Initialize $\tilde{X}_0 \equiv X_0 \equiv \log 75$ (\$75 per share), and then iterate:

$$\tilde{X}_i = \tilde{X}_{i-1} + \mu\Delta_n + \sigma\sqrt{\Delta_n} Z_i \text{ for } i = 1, 2, \dots, nT$$

After computing all $\tilde{X}_i$'s, you can convert back to prices by computing $P_i \equiv \mathrm{e}^{\tilde{X}_i}$.

1.2 The Jump Part

To simulate from the jump process, first define the intensity of the jumps, say $\lambda = 15/252$ (15 jumps per year on average). Then obtain the total number of jumps by taking a random draw from a Poisson distribution:

$$N \overset{d}{\sim} \text{Poisson}(\lambda T)$$

Now, draw N jumps from a normal distribution with variance σ^2_{jump}. For example, a value of $\sigma_{jump} = 18 \times \sqrt{\sigma^2/n}$ implies jumps have 18 times the volatility of the diffusive moves in the asset (18 is just for illustration).

$$Y_k \overset{d}{\sim} \mathcal{N}(0, \sigma^2_{jump}) \text{ for } k = 1, 2, \ldots, N$$

To distribute the jumps at random times, draw N independent random numbers from the uniform distribution:

$$U_k \overset{d}{\sim} \text{Unif}(0, 1) \text{ for } k = 1, 2, \ldots, N$$

Notice that we changed the distribution of U_k from the last lecture notes. Previously, we had $U_k \overset{d}{\sim} \text{Unif}(0, T)$, which gave us the random times across the interval $[0, T]$. The change to Unif$(0, 1)$ is innocuous and helpful, since most statistical packages provide a function for generating uniform random values in the $(0, 1)$ interval. Notice that if $U_k \overset{d}{\sim} \text{Unif}(0, 1)$, then $T \cdot U_k \overset{d}{\sim} \text{Unif}(0, T)$.

Now, form the discretized jump process:

$$J_i = \sum_{k=1}^{N} \mathbb{1}_{\left\{U_k \leq \frac{i}{nT}\right\}} Y_k \text{ for } i = 0, 1, 2, \ldots, nT$$

Notice that the indicator can be re-written as:

$$\begin{aligned} \mathbb{1}_{\left\{U_k \leq \frac{i}{nT}\right\}} &= \mathbb{1}_{\left\{T \cdot U_k \leq \frac{i}{n}\right\}} \\ &= \mathbb{1}_{\{T \cdot U_k \leq i \cdot \Delta_n\}} \end{aligned}$$

Thus, the indicator is checking whether the realization of $T \cdot U_k$ (which is uniformly distributed over $[0, T]$) is smaller than the position of the i-th interval.

1.3 Combining the Two Parts

Combine the simulation of the continuous part ($\tilde{X}_i$'s) with the simulation of the jump process (J_i's) to get the simulation for the log-price process:

$$X_i = \tilde{X}_i + J_i \text{ for } i = 1, 2, \ldots, nT$$

You can convert from log-prices to prices by computing $P_i \equiv \mathrm{e}^{X_i}$.

2 Stochastic Variance

Now, we want to simulate the Jump-Diffusion when the variance is time varying and random. To simplify things, we will ignore the jump part of the process (you can always add it back) and focus only on the variance.

We will simulate the model:

$$dX_t = \sqrt{c_t} dW_t$$
$$dc_t = \rho(\mu_c - c_t)dt + \sigma_c \sqrt{c_t} dW_t^c$$

where we assume W and W^c are independent (for simplicity).

Fix $n = 80$, so that $\Delta_n = 0.0125$, and let $T = 1.25 \times 252$. Fix values for ρ, μ_c and σ_c.

To simulate this model, we need to use an Euler discretization scheme. This discretization scheme means we will simulate the model at a very high frequency, and then sample at a coarser frequency to mimic the actual observed data. This is necessary to minimize the discretization error within the simulation.

Let the number of Euler discretization intervals be given by $n_E = 20 \times n$, so that the discretization interval size is $\delta_E \equiv \frac{1}{n_E} < \Delta_n$.

Observe that a value of $n = 80$ is equivalent to 80 5-min observations per day. A value of $n_E = 20 \times n$ is equivalent to sampling every 15 seconds.

To simulate the variance process, first generate $n_E T$ independent draws from the standard normal distribution:

$$\tilde{Z}_j \overset{d}{\sim} \mathcal{N}(0,1) \text{ for } j = 1, 2, \ldots, n_E T$$

Then, given an initial value for the variance ($c_0 \equiv \mu_c$) we can iterate the variance process:

$$c_j = c_{j-1} + \rho(\mu_c - c_{j-1})\delta_E + \sigma_c \sqrt{c_{j-1}} \sqrt{\delta_E} \tilde{Z}_j \text{ for } j = 1, 2, \ldots, n_E T$$

Because of the discretization there is a small chance that the simulated c_j could go negative. When that happens, reflect the process by setting $c_j = \mu_c/2$.

Given the simulation for the variance process, we can simulate the log-price. First, generate $n_E T$ independent draws from the standard normal distribution:

$$Z_j \overset{d}{\sim} \mathcal{N}(0,1) \text{ for } j = 1, 2, \ldots, n_E T$$

The Z_j's here are fresh draws from the standard normal and are independent of the $\tilde{Z}_j$'s from the variance process.

Now, given $\tilde{X}_0 \equiv X_0 \equiv \log 75$, we can iterate the process:

$$\tilde{X}_j \equiv \tilde{X}_{j-1} + \sqrt{\tilde{c}_{j-1}} \sqrt{\delta_E} Z_j \text{ for } j = 1, 2, \ldots, n_E T$$

Finally, sample the $\tilde{X}_j$'s at a coarser frequency:

$$X_i \equiv \tilde{X}_{\left(i \frac{n_E}{n}\right)} \text{ for } i = 0, 1, 2, \ldots, nT$$