Midterm Exam Solutions

The purpose of these notes is not to be a comprehensive solutions manual, but to offer guidance on how to solve some of the exam questions.

1 Question 1

This question is about estimating the value at risk for a position on some stock. We ignore rare jumps to simplify the problem.

We know that the return of the stock over a day will be given by the diffusive part:

$$r_t^c = \int_{t-1}^{t} \sqrt{c_s} dW_s \approx \sqrt{\int_{t-1}^{t} c_s ds} Z_t = \sqrt{\text{IV}_t} Z_t \overset{d}{\sim} \mathcal{N}(0, \text{IV}_t)$$

The issue is that we cannot estimate IV_t for the day we are interested in (we do not know the returns before they actually happen). We then assume yesterday's integrated variance is a perfect forecast for today's integrated variance:

$$r_t^c \overset{d}{\sim} \mathcal{N}(0, \text{IV}_{t-1})$$

Under this assumption, we can estimate IV_{t-1} using the truncated variance estimator and yesterday's returns. For a small enough Δ_n:

$$\frac{r_t^c}{\sqrt{\text{TV}_{t-1}}} \overset{d}{\sim} \mathcal{N}(0, 1)$$

To compute the probability of observing a return below $-q$, notice that:

$$\mathbb{P}(r_t^c \leq -q) = \mathbb{P}\left(\frac{r_t^c}{\sqrt{\text{IV}_{t-1}}} \leq \frac{-q}{\sqrt{\text{IV}_{t-1}}}\right) = \Phi\left(\frac{-q}{\sqrt{\text{IV}_{t-1}}}\right)$$

Where Φ is the cumulative density function of the standard normal distribution. To estimate the probability above, all we need to do is plug-in an estimator for IV_{t-1}:

$$\hat{\mathbb{P}}(r_t^c \leq -q) = \Phi\left(\frac{-q}{\sqrt{\text{TV}_{t-1}}}\right)$$

Now, to obtain an approximate confidence interval for the probability above, one could simply compute the bounds of the confidence interval for IV_{t-1}, and then plug those in the equation above:

$$\text{CI}(\text{IV}_{t-1}, \alpha) = \left[\underbrace{\text{TV}_{t-1} - \Phi^{-1}(1-\alpha/2)\sqrt{2\Delta_n \widehat{\text{QIV}_{t-1}}}}_{\text{IV}_{lower}}, \underbrace{\text{TV}_{t-1} + \Phi^{-1}(1-\alpha/2)\sqrt{2\Delta_n \widehat{\text{QIV}_{t-1}}}}_{\text{IV}_{upper}}\right]$$

Using these values:

$$\text{CI}\left(\mathbb{P}\left(r_t^c \leq -q\right), \alpha\right) \approx \left[\Phi\left(\frac{-q}{\sqrt{\text{IV}_{lower}}}\right), \Phi\left(\frac{-q}{\sqrt{\text{IV}_{upper}}}\right)\right]$$

This is would be an accepted solution, provided you implemented everything correctly in Matlab. An alternative would be to use the Delta-method to derive the confidence interval (see the last section).

To compute the Value at Risk, we can use a similar approach:

$$\begin{aligned}\mathbb{P}\left(r_t^c \times V \leq Q\right) = p &\iff \mathbb{P}\left(\frac{r_t^c}{\sqrt{\text{IV}_{t-1}}} \leq \frac{Q}{V\sqrt{\text{IV}_{t-1}}}\right) = p \\ &\iff \Phi\left(\frac{Q}{V\sqrt{\text{IV}_{t-1}}}\right) = p \\ &\iff Q = V\sqrt{\text{IV}_{t-1}}\Phi^{-1}(p)\end{aligned}$$

To estimate the Value at Risk, all we need is to plug-in an estimator for IV_{t-1}:

$$\hat{Q} = V\sqrt{\text{TV}_{t-1}}\Phi^{-1}(p)$$

Once again, to obtain an approximate confidence interval for Q, one could simply plug-in IV_{lower} and IV_{upper} in the equation above:

$$\text{CI}(Q, \alpha) \approx \left[V\sqrt{\text{IV}_{upper}}\Phi^{-1}(p), V\sqrt{\text{IV}_{lower}}\Phi^{-1}(p)\right]$$

We switch the order of IV_{lower} and IV_{upper} because for small values of p, the number $\Phi^{-1}(p)$ is negative.

This is would be an accepted solution, provided you implemented everything correctly in Matlab. An alternative solution would be to use the Delta-method to derive the confidence interval (see the last section).

2 Question 2

This question is asking you to evaluate whether a portfolio is market-neutral. We do so by estimating the realized beta between the portfolio and the market and verifying if there is evidence the realized betas are all zero.

To estimate the realized betas you need the diffusive returns of the market index and of the long-short portfolio. There are two acceptable ways of obtaining the diffusive returns of the portfolio:

1. Obtain the diffusive returns of each stock separately, and then subtract one from another to obtain the diffusive returns of the portfolio
2. Subtract the total returns to obtain the returns of the portfolio, then use the jump detector to obtain the diffusive returns of the portfolio

The remainder of the question requires estimating the realized beta and computing confidence intervals based on the bootstrap method. To run the bootstrap, you need to re-sample the diffusive returns of the long-short portfolio and of the market index. Because there is an intraday pattern in the volatility, we need to obtain the new random sample using small blocks of size k_n. Additionally, because the portfolio and the market index might co-move, we need to re-sample both the portfolio and the market index at the same time (using the same random numbers on both assets).

3 Question 3

This question is interested in analyzing whether there is idiosyncratic co-movement between stocks. To do so, we remove the effect of stock co-movements with market index from the stock returns, obtaining the stock's idiosyncratic returns. Then, we estimate the correlation between the idiosyncratic returns of two stocks and check if there is evidence on whether there is or there isn't correlation.

To estimate the correlation between the idiosyncratic returns of two stocks, we use the following residuals:

$$\hat{e}_{1,t,i} = r^c_{1,t,i} - \widehat{R\beta}_{1,t} r^c_{\mathrm{SPY},t,i}$$
$$\hat{e}_{2,t,i} = r^c_{2,t,i} - \widehat{R\beta}_{2,t} r^c_{\mathrm{SPY},t,i}$$

Where the realized beta estimates above are with respect to the market index returns. Then, the correlation is simply $\hat{\rho}_{e,t} = \mathrm{Corr}(\hat{e}_{1,t,i}, \hat{e}_{2,t,i})$.

To obtain confidence intervals for the correlations, we use the bootstrap method. In this case, we resample the estimated residuals, taking into account the intraday patterns and possible co-movements between the two residuals. After obtaining a new random sample, we can compute an estimate of the correlation. We then repeat the above, and find the confidence intervals.

4 Delta-Method on Question 1

On the first part of Question 1, we want to compute the probability of observing a return worse than some value $-q$. We found the expression:

$$\mathbb{P}\left(r^c_t \leq -q\right) = \Phi\left(\frac{-q}{\sqrt{\mathrm{IV}_{t-1}}}\right)$$

We also know the asymptotic distribution of the truncated variance:

$$\Delta_n^{-\frac{1}{2}}\left(\mathrm{TV}_{t-1} - \mathrm{IV}_{t-1}\right) \xrightarrow{d} \mathcal{N}\left(0, 2\mathrm{QIV}_{t-1}\right)$$

We can apply the Delta method to obtain the asymptotic distribution of the probability of interest. Consider the function:

$$g(x) \equiv \Phi\left(\frac{-q}{\sqrt{x}}\right)$$

This function is defined for values $x > 0$, which is inline with our case, since the integrated variance is naturally always positive. Also notice that g is continuous, since Φ is continuous. Therefore:

$$g'(x) = \frac{qx^{-3/2}}{2}\Phi'(-qx^{-1/2})$$
$$= \frac{qx^{-3/2}}{2} \cdot \frac{\mathrm{e}^{-\frac{q^2}{2x}}}{\sqrt{2\pi}}$$

Applying the Delta-method with function g above on the asymptotic distribution of the truncated variance yields:

$$\Delta_n^{-\frac{1}{2}}\left(\hat{\mathbb{P}}\left(r^c_t \leq -q\right) - \mathbb{P}\left(r^c_t \leq -q\right)\right) \xrightarrow{d} \mathcal{N}\left(0, 2\mathrm{QIV}_{t-1} g'(\mathrm{IV}_{t-1})^2\right)$$

We can estimate the asymptotic variance by plugging in estimators for QIV_{t-1} and IV_{t-1}:

$$\widehat{\text{AVar}} = 2\widehat{\text{QIV}}_{t-1} g'(\text{TV}_{t-1})^2$$

And the confidence interval can be created in the usual way.

On the second part of Question 1, we want to compute confidence intervals for the value at risk. We derived the expression:

$$Q = V\sqrt{\text{IV}_{t-1}}\Phi^{-1}(p)$$

Notice that we can also the Delta-method here to obtain confidence intervals for Q. This time, consider the function h as below:

$$h(x) \equiv V\Phi^{-1}(p)\sqrt{x}$$

where V and $\Phi^{-1}(p)$ are just constants. Then:

$$h'(x) = \frac{V\Phi^{-1}(p)}{2\sqrt{x}}$$

Applying the Delta-method with function h to the asymptotic distribution of the truncated variance yields:

$$\Delta_n^{-\frac{1}{2}}\left(\hat{Q} - Q\right) \xrightarrow{d} \mathcal{N}\left(0, 2\text{QIV}_{t-1} h'(\text{IV}_{t-1})^2\right)$$

We can estimate the asymptotic variance by plugging in estimators for QIV_{t-1} and IV_{t-1}:

$$\widehat{\text{AVar}} = 2\widehat{\text{QIV}}_{t-1} h'(\text{TV}_{t-1})^2$$

And the confidence interval can be created in the usual way.