Delta Method

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The Delta Method is a useful result that allows us to quickly change the units of measurement of an estimator and adapt its limiting distribution.

Suppose we have an estimator $\hat{\theta}_n$ of some statistic we care about θ. Additionally, suppose we know its limiting distribution:

$$\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{d} \mathcal{N}\left(0, \sigma^2\right)$$

Given a function $g : \mathbb{R} \to \mathbb{R}$ that is differentiable at θ, then:

$$\sqrt{n}(g(\hat{\theta}_n) - g(\theta)) \xrightarrow{d} \mathcal{N}\left(0, \sigma^2(g'(\theta))^2\right)$$

That is, by rescaling the estimator with a smooth function we only change the variance of the asymptotic distribution.

We can use the Delta Method to obtain the asymptotic distribution of annualized versions of the estimators we have studied so far.

For example, if there are no jumps:

$$\Delta_n^{-\frac{1}{2}}(RV_t - IV_t) \xrightarrow{d} \mathcal{N}\left(0, 2\int_{t-1}^{t} c_s^2 ds\right)$$

Notice that:

$$\begin{aligned}\Delta_n^{-\frac{1}{2}} &= \left(\frac{1}{n}\right)^{-\frac{1}{2}} \\ &= \sqrt{n}\end{aligned}$$

Define the functions:

$$g : x \mapsto \sqrt{252 \times x}$$

Then the asymptotic distribution of the annualized RV is:

$$\Delta_n^{-\frac{1}{2}}(\sqrt{252 RV_t} - \sqrt{252 IV_t}) \xrightarrow{d} \mathcal{N}\left(0, 2(g'(IV_t))^2 \int_{t-1}^{t} c_s^2 ds\right)$$

Using the notation from last lecture:

$$\Delta_n^{-\frac{1}{2}}(\sqrt{252RV_t} - \sqrt{252IV_t}) \xrightarrow{d} \mathcal{N}\left(0, 2(g'(IV_t))^2 QIV_t\right)$$

Plugging in estimators for the new asymptotic variance we get:

$$\Delta_n^{-\frac{1}{2}} \frac{\sqrt{252RV_t} - \sqrt{252IV_t}}{\sqrt{2\widehat{QIV}_t}\; g'(TV_t)} \xrightarrow{d} \mathcal{N}(0,1)$$

We can use the asymptotic distribution to create confidence intervals for the annualized integrated variance.