Jump-Diffusion Processes

Guilherme Salomé

August 30, 2018

1 Brownian Semimartingales

We consider a continuous time processes X_t for $0 \leq t \leq T$. The unit of time (unit of t and T) is always one day. Usually T is an integer equal to 1 (a day), 22 (a business month) or 252 (a business year).

A semimartingale is a càdlàg process (right-continuous with left limits) that consists of a sum of two processes: one of finite variation, and one that is a local martingale.

$$X_t = X_0 + A_t + M_t$$

A first example of a semimartingale is the Gaussian diffusion with constant coefficients.

1.1 Gaussian Diffusion with Constant Coefficients

A Gaussian diffusion with constant coefficients is given by:

$$X_t = at + \sqrt{c}W_t$$

where a and $c > 0$ are constants and W is a Wiener process.

Remember that a Wiener process satisfies $W_0 = 0$ and has increments that are conditionally independent and normally distributed, so that $\mathbb{E}[W_t - W_s | W_s] = 0$ for $s \leq t$ and $\text{Var}\left[W_t\right] = t$.

The Gaussian diffusion with constant coefficients model is a very natural representation of a financial times series. Indeed, the term $a\tau$ represents the local predictable risk premium earned over the interval $t \to t + \tau$ for bearing the risk of the outcome $\sqrt{c}W_{t+\tau} - \sqrt{c}\, W_t$.

The easiest way to understand this model is to think about how to simulate it given values for the constants a and c. Suppose the discretization interval is Δ_n (a very small number) and that we want $n = 1/\Delta_n$ steps per day, over T days in total.

Given $X_0 = 0$, the next step is given by:

$$\begin{aligned} X_{\Delta_n} &= a\Delta_n + \sqrt{c}W_{\Delta_n} \\ &= a\Delta_n + \sqrt{c}\underbrace{\left(W_{\Delta_n} - W_0\right)}_{\overset{d}{\sim}\mathcal{N}(0,\Delta_n)} \end{aligned}$$

So the first step can be generated by generating a random value from the normal distribution.

Let's do one more step:

$$\begin{aligned} X_{2\Delta_n} &= a(2\Delta_n) + \sqrt{c}W_{2\Delta_n} \\ &= a\Delta_n + a\Delta_n + \sqrt{c}\left(W_{2\Delta_n} - W_{\Delta_n} + W_{\Delta_n} - W_0\right) \\ &= X_{\Delta_n} + a\Delta_n + \sqrt{c}\underbrace{\left(W_{2\Delta_n} - W_{\Delta_n}\right)}_{\overset{d}{\sim}\mathcal{N}(0,\Delta_n)} \end{aligned}$$

That is, to generate the next step of the simulation, all we need is the value of the first step, and another random value from the normal distribution.

In general, to simulate the process for times $t = 0, \ldots, T$ generate nT independent and identically distributed draws from the standard normal (denoted by Z_i), and set:

$$X_t^n = \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \left(a\Delta_n + \sqrt{c}\sqrt{\Delta_n}Z_i\right) \quad \text{for } t \in [0, T]$$

The superscript n on X_t^n indicates that we have n steps for each day, each of size Δ_n. We have $\mathbb{E}[X_1^n] = a, \operatorname{Var}[X_1^n] = c$, and $\mathbb{E}[X_T^n] = aT, \operatorname{Var}[X_T^n] = cT$. Jacod and Protter (2012) and many others have shown that $X^n \to X$ as a stochastic process as $\Delta_n \to 0$.

1.2 Gaussian Diffusion with Variable Coefficients

The model with variable coefficients is similar to the previous one, but it allows the local drift and the variance to be random and time varying.

Let $\{a_s, c_s\}_{s\in[0,T]}$ be left continuous and independent of the Wiener process. Then, we can write the model as:

$$X_t = \int_0^t a_s ds + \int_0^t \sqrt{c_s} dW_s$$

The first integral is an ordinary integral and the second is a stochastic integral.

We can also simulate this process as before. Starting from $X_0 = 0$, we can write the first step as:

$$X_{\Delta_n} = a_{\Delta_n}\Delta_n + \sqrt{c_{\Delta_n}}\underbrace{\left(W_{\Delta_n} - W_0\right)}_{=\sqrt{\Delta_n}Z_1}$$

Now a_s, c_s are time varying, so the simulation also requires the discretization of these processes.

The value of the simulated process at time t is given by:

$$X_t^n = \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \left(a_{i/n}\Delta_n + \sqrt{c_{i/n}}\sqrt{\Delta_n}Z_i\right) \quad \text{for } t \in [0, T]$$

In Jacod and Protter, 2012 it is shown that $X^n \to X$.

2 Compound Poisson Processes

Compound Poisson processes for $t \in [0, T]$ are fundamental to Lévy processes and financial econometrics. Contrary to the diffusion processes discussed previously, a compound

Poisson process is a process driven by jumps. The jumps in this process arrive at random times, and the size of the jumps follow some distribution.

The number of jumps in the process is given by a Poisson distribution. Let $\lambda > 0$ and $N \sim \text{Poisson}(\lambda T)$. N represents the total number of jumps on the interval $[0, T]$. The size of the jumps are determined independently of N and are given by the i.i.d. random variables $\{X_i\}_{i=1}^N$. Conditioned on the number of jumps (N), all that needs to be done is to scatter the N jumps uniformly over $[0, T]$. To this end, Let $\{U_i\}_{i=1}^N$ be independent Unif$[0, T]$ random variables, and set:

$$Y_t = \sum_{i=1}^{N} 1_{\{U_i \leq t\}} X_i \quad \text{for } 0 \leq t \leq T$$

Then Y_t is a compound Poisson process with intensity parameter λ and jump density $f(x)$. The characteristic function of Y_1 is:

$$\mathbb{E}\left[e^{iuY_1}\right] = e^{\lambda \int (e^{iux}-1) f(x) dx}$$

and that of Y_t is

$$\mathbb{E}\left[e^{iuY_t}\right] = e^{t \int (e^{iux}-1) m(x) dx} \quad \text{for } t \geq 0$$

where $m(x) \equiv \lambda f(x)$ is known as the intensity density. m receives this name because it integrates to the jump intensity:

$$\int m(x) dx = \lambda < \infty$$

It also gives the expected number of jumps in Y_t of size between a and b:

$$t \int_a^b m(x) dx.$$

We can compute the relative frequency of jumps in $[a, b]$ relative to $[c, d]$ as the ratio

$$0 \leq \frac{\int_a^b m(x)dx}{\int_c^d m(x)dx} < \infty$$

Note that jumps of size 0 make no sense so we can redefine the jump intensity at 0 as $m(0) = 0$. If $a < 0$ and $b > 0$ then strictly speaking we should write the expected number of jumps of Y_t in the interval $[a, b]$ is

$$t \int_a^{0^-} m(x)dx + t \int_{0^+}^b m(x)dx$$

which is tedious so we will just write this as

$$t \int_a^b m(x)dx$$

with the understanding that the integral above skips over the point $x = 0$. We will see later that the behavior of the intensity function around 0, i.e., for very small positive or negative jumps is delicate and very important for studying a Lévy process.

3 Jump-Diffusion Processes

A Jump-Diffusion process is a process that has a diffusion component and a jump component:

$$X_t = \int_0^t a_s ds + \int_0^t \sqrt{c_s} dW_s + \tilde{Y}_t \quad \text{for } 0 \leq t \leq T.$$

As before, the first integral is an ordinary integral, the second is a stochastic integral, and $\tilde{Y}_t = Y_t - t\lambda\mu_x$ is a centered compound Poison process with intensity density $m(x)$; centering makes the a_t process be the true drift. X_t is a semimartingale and is the most commonly used models for financial prices. Note that with $\Delta X_t \equiv X_t - X_{t-}$

$$X_t' = X_t - \Delta X_t$$

is continuous and thereby called the continuous part of X. We often think of the decomposition

$$X_t = X_t' + \Delta X_t$$

4 Sampling

Now let's suppose the Jump-Diffusion process X_t is well defined and consider discrete equi-spaced observations at sampling interval Δ_n on it: $X_0, X_{\Delta_n}, X_{2\Delta_n}, \ldots, X_{nT\Delta_n}$, where $n = [1/\Delta_n]$. We always work with the increments

$$\Delta_i^n X = X_{i\Delta_n} - X_{(i-1)\Delta_n}, \quad i = 1, 2, \ldots, nT.$$

In the next lectures we will define estimators for variance based on sums of $|\Delta_i^n X|^2$ and $|\Delta_i^n X||\Delta_{i-1}^n X|$.

5 The 3 Uses of Δ in High-Frequency Financial Econometrics

1. The jump operator: $\Delta X_t = X_t - X_{t-}$
2. The width of the sampling interval: Δ_n, $\Delta_n = \frac{1}{n}$, or $n = \lfloor \frac{1}{\Delta_n} \rfloor$
3. The first difference operator: $\Delta_i^n X = X_{i\Delta_n} - X_{(i-1)\Delta_n)}$

The meaning is deduced from the context.

References

Jacod, Jean and Philip Protter (2012). *Discretization of Processes*. Springer-Verlag.